Imperial College
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Lecture 17

Digital Filters
(Lathi 5.6)

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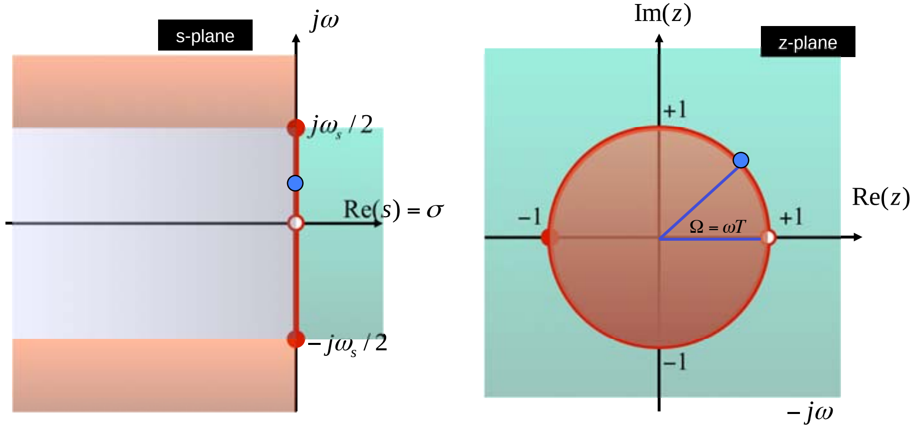
Mapping from s-plane to z-plane (from last lecture)

- Since $z = e^{sT} = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T}$ where T = $2\pi/\omega_s$
we can map the s-plane to the z-plane as below:

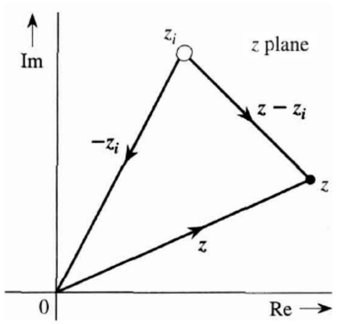


Frequency Response from pole-zero locations (1)

- The transfer function H[z] can be expressed in factorised polynomial:

$$H[z] = b_0 \frac{(z - z_1)(z - z_2)\cdots(z - z_N)}{(z - \gamma_1)(z - \gamma_2)\cdots(z - \gamma_N)}$$

- We have established that the frequency response is given by $H[e^{j\Omega}]$.
- Therefore we can compute the frequency be evaluating H[z] at z=$e^{j\Omega}$, which is the unity circle.
- Each term (z-z_i) can be evaluated as shown:



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Frequency Response from pole-zero locations (2)

- Therefore, for all the poles and zeros, we can use the graphical method (similar to the s-plane case):
- The amplitude response is:

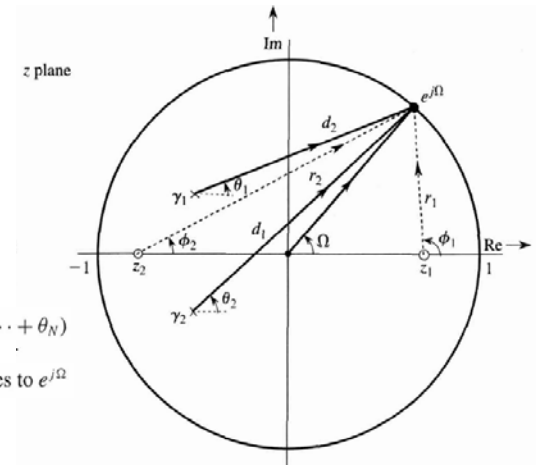
$$|H[e^{j\Omega}]| = b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N}$$

$$= b_0 \frac{\text{product of the distances of zeros to } e^{j\Omega}}{\text{product of distances of poles to } e^{j\Omega}}$$

- The phase response is:

$$\angle H[e^{j\Omega}] = (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N)$$

$$= \text{sum of zero angles to } e^{j\Omega} - \text{sum of pole angles to } e^{j\Omega}$$



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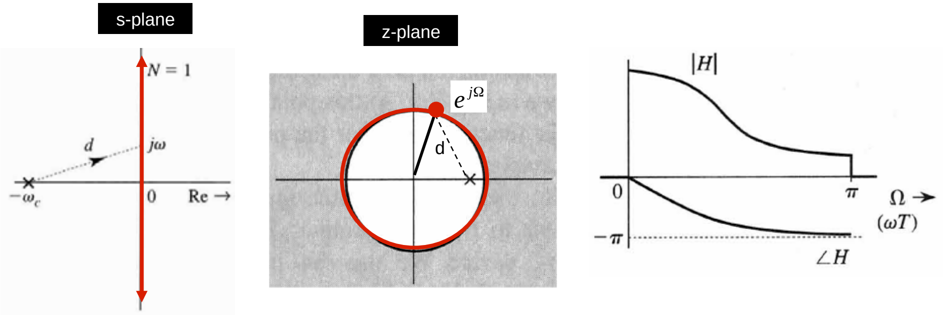
Pole-Zero locations & Filtering (1)

- Therefore, for all the poles and zeros, we can use the graphical method (similar to the s-plane case):



Evaluate H(jω)=1/d

Evaluate $H[e^{j\Omega}]=1/d$

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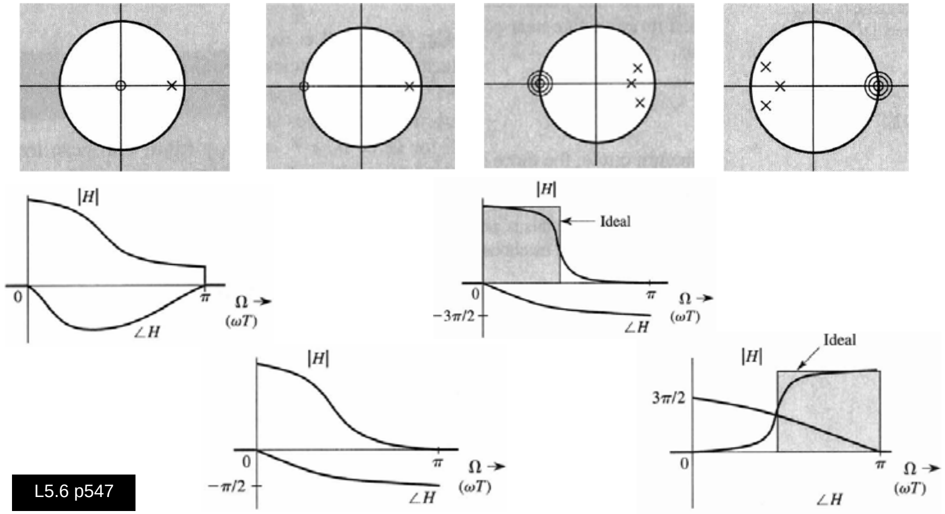
Pole-Zero locations & Filtering (2)



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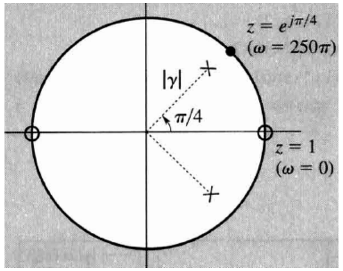
Example – a bandpass filter (1)

- Derive the amplitude response of a discrete-time system having a sampling frequency of 1000 Hz, and with zeros at (z=1) and (z=0), and poles at $\gamma_1 = |\gamma| e^{j\pi/4}$ and $\gamma_2 = |\gamma| e^{-j\pi/4}$.
- The transfer function is:

$$H[z] = \frac{(z-1)(z+1)}{(z-|\gamma|e^{j\pi/4})(z-|\gamma|e^{-j\pi/4})} = \frac{z^2-1}{z^2-\sqrt{2}|\gamma|z+|\gamma|^2}$$



- Note that analogue frequency ω corresponds to digital frequency Ω= ωT, where T is the sampling period (i.e. 10^{-3}).
- Hence Ω=π/4 corresponds to ω=250π or *f* =125Hz.

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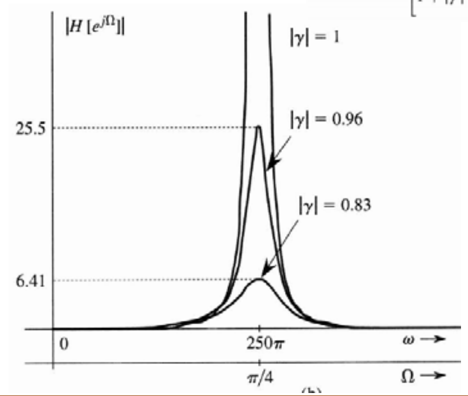
Example – a bandpass filter (2)

- Therefore, we have:

$$|H[e^{j\Omega}]| = \frac{|e^{j2\Omega}-1|}{|e^{j\Omega}-|\gamma|e^{j\pi/4}||e^{j\Omega}-|\gamma|e^{-j\pi/4}|}$$

$$|H[e^{j\Omega}]|^2 = \frac{2(1-\cos 2\Omega)}{\left[1+|\gamma|^2-2|\gamma|\cos\left(\Omega-\frac{\pi}{4}\right)\right]\left[1+|\gamma|^2-2|\gamma|\cos\left(\Omega+\frac{\pi}{4}\right)\right]}$$



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Example – a bandpass filter (3)

- We can implement this digital filter as:

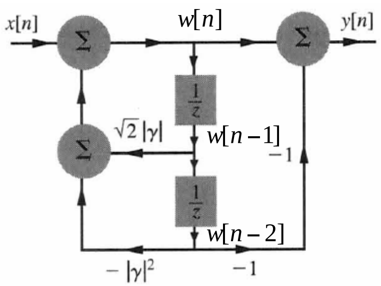
$$H[z] = \frac{z^2 - 1}{z^2 - \sqrt{2}|\gamma|z + |\gamma|^2} \qquad \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 - \sqrt{2}|\gamma|z^{-1} + |\gamma|^2 z^{-2}}$$

$$Y(z) = \left(1 - z^{-2}\right)\frac{1}{1 - \sqrt{2}|\gamma|z^{-1} + |\gamma|^2 z^{-2}} X(z)$$

$$Y(z) = \left(1 - z^{-2}\right)W(z)$$

$$W(z) = \frac{1}{1 - \sqrt{2}|\gamma|z^{-1} + |\gamma|^2 z^{-2}} X(z)$$

$$W(z) = X(z) + \sqrt{2}|\gamma|z^{-1}W(z) - |\gamma|^2 z^{-2}W(z)$$



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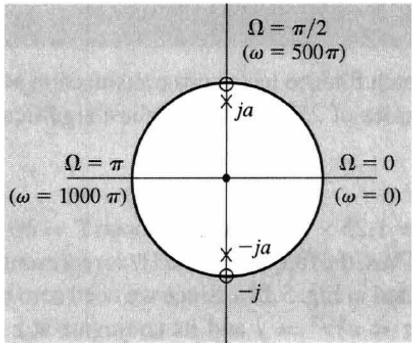
Example – a bandstop (notch) filter (1)

- Design a second-order notch filter to have zero transmission (i.e notch) at 250Hz. Assume sample frequency to be 1000Hz.
- One revolution around unity circuit corresponds to 1000Hz, the sampling frequency range.
- Therefore 250Hz corresponds to $\Omega=2\pi x250/1000=\pi/2$.
- Therefore place two zeros on unity circle at $\Omega=\pm\pi/2$ to form the notch.
- To make the recovery around the notch frequency "fast", place two poles as shown at distance a from origin.

$$H[z] = \frac{(z-j)(z+j)}{(z-ja)(z+ja)} = \frac{z^2+1}{z^2+a^2}$$



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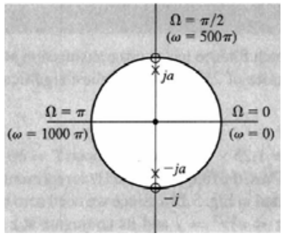
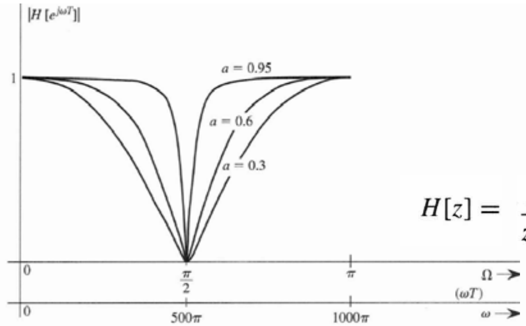
Example – a bandstop (notch) filter (2)

- Therefore, the square of the magnitude response is:

$$|H[e^{j\Omega}]|^2 = \frac{(1+a^2)^2}{4} \frac{(e^{j2\Omega}+1)(e^{-j2\Omega}+1)}{(e^{j2\Omega}+a^2)(e^{-j2\Omega}+a^2)}$$

$$= \frac{(1+a^2)^2(1+\cos 2\Omega)}{2(1+a^4+2a^2\cos 2\Omega)}$$





$$H[z] = \frac{z^2+1}{z^2+a^2}$$